Proof of a congruence on sums of powers of q-binomial coefficients

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Abstract. We prove that, if $m, n \geqslant 1$ and $a_1, \ldots, a_m$ are nonnegative integers, then

$$\frac{[a_1+\cdots+a_m+1]!}{[a_1]!\ldots[a_m]!}\sum_{h=0}^{n-1} q^h \prod_{i=1}^{m} \begin{bmatrix} h \\ a_i \end{bmatrix} \equiv 0 \pmod{[n]},$$

where $[n]=\frac{1-q^n}{1-q}$, $[n]!=[n][n-1]\cdots[1]$, and $\begin{bmatrix} a \\ b \end{bmatrix}=\prod_{k=1}^{b}\frac{1-q^{a-k+1}}{1-q^k}$. The $a_1=\cdots=a_m$ case confirms a recent conjecture of Z.-W. Sun. We also show that, if p is a prime greater than $\max\{a,b\}$, then

$$\frac{[a+b+1]!}{[a]![b]!}\sum_{h=0}^{p-1} q^h \begin{bmatrix} h \\ a \end{bmatrix}\begin{bmatrix} h \\ b \end{bmatrix} \equiv (-1)^{a-b} q^{ab-\binom{a}{2}-\binom{b}{2}}[p] \pmod{[p]^2}.$$

Keywords: q-binomial coefficients; q-Chu-Vandermonde summation; q-Pfaff-Saalschütz's identity; Faulhaber's formula

MR Subject Classifications: 11A07, 11B65, 05A10

1 Introduction

Recall that the *q-binomial coefficients* $\begin{bmatrix} n \\ k \end{bmatrix}$ (see [2]) are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}, & \text{if } 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

q-Binomial coefficients are closely related to binomial coefficients by the relation $\lim_{q\to 1}\begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k}$. Recently, Z.-W. Sun [6] proved many interesting congruences on sums involving binomial coefficients or q-binomial coefficients. For example, Sun [6] proved that, for any nonnegative integers n and k with $n > k$, there holds

$$[2k+1]\begin{bmatrix} 2k \\ k \end{bmatrix}\sum_{h=0}^{n-1} q^h \begin{bmatrix} h \\ k \end{bmatrix}^2 \equiv 0 \pmod{[n]}, \tag{1.1}$$

where $[n] := 1 + q + \cdots + q^{n-1}$, and so

$$(2k+1)\binom{2k}{k}\sum_{h=0}^{n-1}\binom{h}{k}^2 \equiv 0 \pmod{n}. \tag{1.2}$$

He also made the following conjecture, which is a generalization of (1.1) and (1.2).

Conjecture 1.1 [6, Conjecture 5.8] *Let m and n be positive integers, and let $0 \leqslant k < n$. Then*

$$\frac{[km+1]!}{([k]!)^m}\sum_{h=0}^{n-1} q^h \begin{bmatrix} h \\ k \end{bmatrix}^m \equiv 0 \pmod{[n]}, \tag{1.3}$$

where $[n]! = [n][n-1]\cdots[1]$, and so

$$\frac{(km+1)!}{(k!)^m}\sum_{h=0}^{n-1}\binom{h}{k}^m \equiv 0 \pmod{n}. \tag{1.4}$$

Conjecture 1.1 for $m = 1$ is easy, and Sun himself is also able to prove the $m = 3$ case of this conjecture.

The aim of this paper is to prove Conjecture 1.1 for arbitrary m by establishing the following more general form.

Theorem 1.2 *Let $m, n \geqslant 1$, and let $a_1, \ldots, a_m$ be nonnegative integers. Then*

$$\frac{[a_1 + \cdots + a_m + 1]!}{[a_1]!\ldots[a_m]!}\sum_{h=0}^{n-1} q^h \prod_{i=1}^{m}\begin{bmatrix} h \\ a_i \end{bmatrix} \equiv 0 \pmod{[n]}, \tag{1.5}$$

and so

$$\frac{(a_1 + \cdots + a_m + 1)!}{a_1!\cdots a_m!}\sum_{h=0}^{n-1}\prod_{i=1}^{m}\binom{h}{a_i} \equiv 0 \pmod{n}. \tag{1.6}$$

It is clear that, when $a_1 = \cdots = a_m = k$, the congruences (1.5) and (1.6) reduce to (1.3) and (1.4), respectively.

For $m = 2$, we shall prove the the following stronger result.

Theorem 1.3 *Let a and b be nonnegative integers and p a prime with $p > \max\{a, b\}$. Then*

$$\frac{[a+b+1]!}{[a]![b]!}\sum_{h=0}^{p-1} q^h \begin{bmatrix} h \\ a \end{bmatrix}\begin{bmatrix} h \\ b \end{bmatrix} \equiv (-1)^{a-b} q^{ab-\binom{a}{2}-\binom{b}{2}}[p] \pmod{[p]^2}, \tag{1.7}$$

and so

$$\frac{(a+b+1)!}{a!b!}\sum_{h=0}^{p-1}\binom{h}{a}\binom{h}{b} \equiv (-1)^{a-b} p \pmod{p^2}.$$

2 Proof of Theorem 1.2

For any $m, n \geqslant 1$ and nonnegative integers $a_1, \ldots, a_m$, let

$$S_n(a_1, \ldots, a_m) = \frac{[a_1 + \cdots + a_m + 1]!}{[n][a_1]! \ldots [a_m]!} \sum_{h=0}^{n-1} q^h \prod_{i=1}^{m} \begin{bmatrix} h \\ a_i \end{bmatrix}.$$

To prove (1.5), it is equivalent to show that $S_n(a_1, \ldots, a_m)$ is a polynomial in q with integer coefficients. By [1, (3.3.9)], we have

$$\sum_{h=0}^{n-1} q^h \begin{bmatrix} h \\ a_1 \end{bmatrix} = \begin{bmatrix} n \\ a_1 + 1 \end{bmatrix} q^{a_1}, \tag{2.1}$$

and so

$$S_n(a_1) = \frac{[a_1 + 1]}{[n]} \begin{bmatrix} n \\ a_1 + 1 \end{bmatrix} q^{a_1} = \begin{bmatrix} n - 1 \\ a_1 \end{bmatrix} q^{a_1}. \tag{2.2}$$

This proves the $m = 1$ case.

For $m \geqslant 2$, by the q-Chu-Vandermonde summation formula (which is equivalent to [1, (3.3.10)])

$$\sum_{k=0}^{n} \begin{bmatrix} a \\ k \end{bmatrix} \begin{bmatrix} b \\ n - k \end{bmatrix} q^{k(b-n+k)} = \begin{bmatrix} a + b \\ n \end{bmatrix}, \tag{2.3}$$

we have

$$\begin{aligned} \begin{bmatrix} h \\ a_{m-1} \end{bmatrix} \begin{bmatrix} h \\ a_m \end{bmatrix} &= \begin{bmatrix} h \\ a_{m-1} \end{bmatrix} \sum_{k=0}^{a_m} \begin{bmatrix} h - a_{m-1} \\ k \end{bmatrix} \begin{bmatrix} a_{m-1} \\ a_m - k \end{bmatrix} q^{k(a_{m-1} - a_m + k)} \\ &= \sum_{k=0}^{a_m} \begin{bmatrix} h \\ a_{m-1} + k \end{bmatrix} \begin{bmatrix} a_{m-1} + k \\ a_m \end{bmatrix} \begin{bmatrix} a_m \\ k \end{bmatrix} q^{k(a_{m-1} - a_m + k)}. \end{aligned}$$

It follows that

$$\begin{aligned} S_n(a_1, \ldots, a_m) = {} & \frac{[a_1 + \cdots + a_m + 1]!}{[n][a_1]! \ldots [a_m]!} \\ & \times \sum_{h=0}^{n-1} q^h \prod_{i=1}^{m-2} \begin{bmatrix} h \\ a_i \end{bmatrix} \sum_{k=0}^{a_m} \begin{bmatrix} h \\ a_{m-1} + k \end{bmatrix} \begin{bmatrix} a_{m-1} + k \\ a_m \end{bmatrix} \begin{bmatrix} a_m \\ k \end{bmatrix} q^{k(a_{m-1} - a_m + k)}. \end{aligned} \tag{2.4}$$

Exchanging the summation order in (2.4), and noticing that

$$\frac{[a_1 + \cdots + a_m + 1]! [a_{m-1} + k]!}{[a_1 + \cdots + a_{m-1} + k + 1]! [a_{m-1}]! [a_m]!} \begin{bmatrix} a_m \\ k \end{bmatrix} = \begin{bmatrix} a_1 + \cdots + a_m + 1 \\ a_m - k \end{bmatrix} \begin{bmatrix} a_{m-1} + k \\ a_{m-1} \end{bmatrix},$$

we obtain the following recurrence relation:

$$\begin{aligned} S_n(a_1, \ldots, a_m) = {} & \sum_{k=0}^{a_m} \begin{bmatrix} a_1 + \cdots + a_m + 1 \\ a_m - k \end{bmatrix} \begin{bmatrix} a_{m-1} + k \\ a_m \end{bmatrix} \begin{bmatrix} a_{m-1} + k \\ a_{m-1} \end{bmatrix} q^{k(a_{m-1} - a_m + k)} \\ & \times S_n(a_1, \ldots, a_{m-2}, a_{m-1} + k). \end{aligned} \tag{2.5}$$

The proof then follows easily by induction on m.

3 Proof of Theorem 1.3

By (2.5) and (2.2), we obtain

$$\frac{[a+b+1]!}{[a]![b]!}\sum_{h=0}^{p-1} q^h \begin{bmatrix} h \\ a \end{bmatrix}\begin{bmatrix} h \\ b \end{bmatrix} = [p]\sum_{k=0}^{b}\begin{bmatrix} a+b+1 \\ b-k \end{bmatrix}\begin{bmatrix} a+k \\ a \end{bmatrix}\begin{bmatrix} a+k \\ b \end{bmatrix}\begin{bmatrix} p-1 \\ a+k \end{bmatrix} q^{k(a-b+k)+a+k}.$$

Noticing that, if $0 \leqslant a+k \leqslant p-1$, then

$$\begin{bmatrix} p-1 \\ a+k \end{bmatrix} = \prod_{i=1}^{a+k}\frac{1-q^{p-i}}{1-q^i} \equiv \prod_{i=1}^{a+k}\frac{1-q^{-i}}{1-q^i} = (-1)^{a+k}q^{-\binom{a+k+1}{2}} \pmod{[p]}.$$

Moreover, since $p > \max\{a,b\}$, we have $\left[\begin{smallmatrix} a+k \\ a \end{smallmatrix}\right] \equiv 0 \pmod{[p]}$ if $a+k \geqslant p$ and $k \leqslant b$. This means that, for $0 \leqslant k \leqslant b$, we always have

$$\begin{bmatrix} a+k \\ a \end{bmatrix}\begin{bmatrix} p-1 \\ a+k \end{bmatrix} \equiv \begin{bmatrix} a+k \\ a \end{bmatrix}(-1)^{a+k}q^{-\binom{a+k+1}{2}} \pmod{[p]}.$$

Therefore, to prove (1.7), it suffices to show that

$$\sum_{k=0}^{b}\begin{bmatrix} a+b+1 \\ b-k \end{bmatrix}\begin{bmatrix} a+k \\ a \end{bmatrix}\begin{bmatrix} a+k \\ b \end{bmatrix}(-1)^k q^{k(a-b+k)+a+k-\binom{a+k+1}{2}} = (-1)^b q^{ab-\binom{a}{2}-\binom{b}{2}}, \tag{3.1}$$

which is just a special case of the q-Pfaff-Saalschütz's identity (see [1, 3.3.12]):

$$\sum_{k=0}^{n}\frac{(x;q)_k(y;q)_k(q^{-n})_k q^k}{(q;q)_k(z;q)_k(xyq^{1-n}/z;q)_k} = \frac{(z/x;q)_n(z/y;q)_n}{(z;q)_n(z/xy;q)_n}, \tag{3.2}$$

where $(a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$.

In fact, replacing (x,y,z,n,k) by $(q^x, q^x, q^{-a-x}, a+b+1, k+1)$ in (3.2), we get

$$\sum_{k=-1}^{a+b}\frac{(q^x;q)_{k+1}(q^x;q)_{k+1}(-1)^{k+1}}{(q^{-a-x};q)_{k+1}(q^{3x-b};q)_{k+1}}\begin{bmatrix} a+b+1 \\ k+1 \end{bmatrix} q^{\binom{k+1}{2}-(k+1)(a+b)}$$
$$= \frac{(q^{-a-2x};q)_{a+b+1}(q^{-a-2x};q)_{a+b+1}}{(q^{-a-x};q)_{a+b+1}(q^{-a-3x};q)_{a+b+1}}. \tag{3.3}$$

It is easy to see that, for $k \geqslant 0$,

$$\lim_{x\to 0}\frac{(q^x;q)_{k+1}}{(q^{-a-x};q)_{k+1}} = (-1)^{a+1}q^{\binom{a+1}{2}}\begin{bmatrix} k \\ a \end{bmatrix},$$

$$\lim_{x\to 0}\frac{(q^x;q)_{k+1}}{(q^{-b+3x};q)_{k+1}} = \frac{(-1)^b}{3}q^{\binom{b+1}{2}}\begin{bmatrix} k \\ b \end{bmatrix},$$

$$\lim_{x\to 0}\frac{(q^{-a-2x};q)_{a+b+1}(q^{-a-2x};q)_{a+b+1}}{(q^{-a-x};q)_{a+b+1}(q^{-a-3x};q)_{a+b+1}} = \frac{4}{3}.$$

Letting $x \to 0$ in (3.3), we are led to

$$\sum_{k=a}^{a+b} \begin{bmatrix} k \\ a \end{bmatrix} \begin{bmatrix} k \\ b \end{bmatrix} \begin{bmatrix} a+b+1 \\ k+1 \end{bmatrix} (-1)^k q^{\binom{k+1}{2}+\binom{a+1}{2}+\binom{b+1}{2}-(k+1)(a+b)} = (-1)^{a-b},$$

which is clearly equivalent to (3.1).

4 An open problem

By Faulhaber's formula (see [3–5]), it is not hard to see that, for positive integers m and n, there holds

$$(2m+2)! \sum_{h=0}^{n-1} h^{2m+1} \equiv 0 \pmod{n^2}.$$

We end this paper with the following conjecture.

Conjecture 4.1 *Let m, n and k be positive integers with $m \geqslant k$. Then*

$$\frac{((2k+1)(2m+1)+1)!}{((2k+1)!)^{2m+1}} \sum_{h=0}^{n-1} \binom{h}{2k+1}^{2m+1} \equiv 0 \pmod{n^2}.$$

Acknowledgments. This work was partially supported by the Fundamental Research Funds for the Central Universities and the National Natural Science Foundation of China (grant 11371144).

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